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Parameter identification problem for the equation of motion of membrane with strong viscosity

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Abstract

The parameter identification problem of constant parameters in the equation of membrane with strong viscosity is studied. The problem is formulated by a minimization of quadratic cost functionals by distributive measurements. The existence of optimal parameters and necessary optimality conditions for the parameters are proved.

Keywords: Identification problem; Equation of membrane with strong viscosity; Quadratic cost functionals; Necessary optimality conditions

1. Introduction

Let Ω be an open bounded set of $\mathbf{R}^n$ with the smooth boundary Γ. The inner product of $\mathbf{R}^n$ is denoted by $x \cdot y$ for $x, y \in \mathbf{R}^n$. We put $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \Gamma$ for $T > 0$. We study an identification problem for diffusive constants α, β in the equation of motion of membrane with strong viscosity

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \alpha \operatorname{div}\left(\dfrac{\nabla y}{\sqrt{1+|\nabla y|^2}}\right) - \beta \Delta \dfrac{\partial y}{\partial t} = f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where f is a forcing function, y_0 and y_1 are initial conditions. In the previous paper [5], we studied the quadratic optimal control problems for (1.1) and established the necessary conditions for the costs of distributive and terminal values observations based on the well-posedness of weak solutions in [4].

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The system (1.1) is proposed to be a model equation of the dynamics of longitudinal motion of vibrating membrane surrounding Ω by taking into accounts of strong (diffusive) viscosity effects (cf. Gilbarg and Trudinger [7]). It is an important problem that whether the physical constants α, β can be estimated or not by the possible observed measurements.

In this paper we study a parameter identification problem of finding best parameters α, β. We explain the setting of this problem. At first, we assume the desired state is known but a constant parameters α, β involved in the equation are unknown. We aim to show the existence of an optimal parameters in an admissible set and its characterizations namely a parameter identification problem in which we use the term optimal parameter to denote the best parameter within any admissible set for which the solution of (1.1) gives a minimum of the given functional.

To obtain our purpose we considered some objective (or penalty) functionals to be minimized on some admissible set. We take this functional by L^2 quadratic norm of observed state minus desired state that is usually regarded as a cost function in optimal control theory.

In this paper we pursue to find necessary conditions for an optimal parameters by using Gâteaux differentiability of the solution mapping and giving variational inequality via an adjoint equation. Proceeding in this way, we can obtain similar results with optimal control problems due to Lions [6], we refer to Ahmed [1] for abstract evolution equations and Ha and Nakagiri [3] for damped Klein–Gordon equations. We take the same approach as in [3].

We explain our identification problem precisely as follows. First we replace the positive constants α, β in (1.1) by $\alpha^2+\alpha_0$, $\beta^2+\beta_0$ in order to take the parameter space $\mathcal{P}=\mathbf{R}^2$ as the set of parameters (α,β) and to guarantee the Gâteaux differentiability of the solution mapping independent of its direction. Let $q=(\alpha,\beta)\in\mathcal{P}$ and let $y(q)=y(q;t,x)$ be the solution for a given $q\in\mathcal{P}$. Let $\mathcal{P}_{\mathrm{ad}}\subset\mathcal{P}$ be an admissible parameter set. We consider the following two quadratic distributive functionals:

$$J_1(q)=\int_0^T\int_\Omega\left|y(q;t,x)-Y_1(t,x)\right|^2dx\,dt\quad\text{for } q\in\mathcal{P},\tag{1.2}$$

$$J_2(q)=\int_0^T\int_\Omega\left|\frac{\partial y(q;t,x)}{\partial t}-Y_2(t,x)\right|^2dx\,dt\quad\text{for } q\in\mathcal{P},\tag{1.3}$$

where $Y_i\in L^2(Q)$, $i=1,2$, are the desired values. We remark that the cost of velocity measurements (1.3) was not treated in [3]. The parameter identification problem for (1.1) with the cost $J=J_1$ in (1.2) or $J=J_2$ in (1.3) is to find and characterize the optimal parameters $q^*=(\alpha^*,\beta^*)\in\mathcal{P}_{\mathrm{ad}}$ satisfying

$$J(q^*)=\inf\{J(q):\ q\in\mathcal{P}_{\mathrm{ad}}\}.\tag{1.4}$$

We prove the existence of an optimal parameter q^* by using the continuity of solutions on parameters and establish the necessary optimality conditions by introducing appropriate adjoint systems. For this we prove the strong Gâteaux differentiability of the nonlinear mapping $q\to y(q)$. We also emphasize that in the velocity's measurements case (1.3), a first order Volterra integro-differential equation is utilized as a proper adjoint system as in [5].

2. Weak solutions and energy equality

For the problem (1.1) we suppose that $f\in L^2(0,T;H^{-1}(\Omega))$, $y_0\in H_0^1(\Omega)$ and $y_1\in L^2(\Omega)$. As in [5] the solution space $W(0,T)$ of (1.1) is defined by

$$W(0,T)=\{g\mid g\in L^2(0,T;H_0^1(\Omega)),\ g'\in L^2(0,T;H_0^1(\Omega)),\ g''\in L^2(0,T;H^{-1}(\Omega))\}$$

endowed with the norm

$$\|g\|_{W(0,T)}=\left(\|g\|^2_{L^2(0,T;H_0^1(\Omega))}+\|g'\|^2_{L^2(0,T;H_0^1(\Omega))}+\|g''\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)^{\frac12}.$$

The space $W(0,T)$ is continuously imbedded in $C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ (cf. Dautray and Lions [2, p. 555]). The scalar products and norms on $L^2(\Omega)$ and $H_0^1(\Omega)$ are denoted by (ϕ,ψ), $|\phi|$ and $(\phi,\psi)_{H_0^1(\Omega)}$, $\|\phi\|$, respectively. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by (ϕ,ψ) and $|\phi|$. Then the scalar product $(\phi,\psi)_{H_0^1(\Omega)}$ and the norm $\|\phi\|$ of $H_0^1(\Omega)$ are given by $(\nabla\phi,\nabla\psi)$ and $\|\phi\| = |(\nabla\phi,\nabla\phi)|^{\frac{1}{2}}$, respectively. The duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle\phi,\psi\rangle$.

The nonlinear operator $G(\nabla\cdot): H_0^1(\Omega) \to [L^2(\Omega)]^n$ is defined by

$$G(\nabla\phi)(x) = \frac{\nabla\phi(x)}{\sqrt{1+|\nabla\phi(x)|^2}} \quad \text{a.e. } x \in \Omega,\ \forall\phi \in H_0^1(\Omega). \tag{2.1}$$

By the definition of $G(\nabla\cdot)$ in (2.1), we have the following property on $G(\nabla\cdot)$:

$$\big|G(\nabla\phi)\big| \leqslant |\nabla\phi|, \qquad \big|G(\nabla\phi) - G(\nabla\psi)\big| \leqslant 2|\nabla\phi - \nabla\psi|, \quad \forall\phi,\psi \in H_0^1(\Omega). \tag{2.2}$$

As in Dautray and Lions [2, p. 555], we give the variational formulation of weak solutions of (1.1). A function y is said to be a weak solution of (1.1) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \big\langle y''(\cdot),\phi\big\rangle + \alpha\big(G\big(\nabla y(\cdot)\big),\nabla\phi\big) + \beta\big(\nabla y'(\cdot),\nabla\phi\big) = \big\langle f(\cdot),\phi\big\rangle \\ \quad \text{for all } \phi \in H_0^1(\Omega) \text{ in the sense of } \mathcal{D}'(0,T), \\ y(0) = y_0 \in H_0^1(\Omega), \quad y'(0) = y_1 \in L^2(\Omega). \end{cases} \tag{2.3}$$

The following two theorems are proved in Hwang and Nakagiri [4] by the Galerkin method.

Theorem 2.1. *Assume that $\alpha,\beta > 0$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega))$. Then the problem* (1.1) *has a unique weak solution y in $W(0,T)$. Furthermore, y has the following estimate*

$$\big|y'(t)\big|^2 + \big|\nabla y(t)\big|^2 + \int_0^t \big|\nabla y'(s)\big|^2\,ds \leqslant C\big(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}\big), \quad \forall t \in [0,T], \tag{2.4}$$

where $C = C(\alpha,\beta)$ is a constant depending only on $\alpha,\beta > 0$.

We will omit writing the integral variables in the definite integral without any confusion. For example, in (2.4) we will write $\int_0^t |\nabla y'|^2\,ds$ instead of $\int_0^t |\nabla y'(s)|^2\,ds$.

Remark 2.1. The constant $C(\alpha,\beta)$ in Theorem 2.1 can be chosen uniformly on any bounded set of (α,β), $\alpha,\beta > 0$, as shown in [4].

3. Identification problems

Let $\mathcal{P} = \mathbf{R}^2$ be the space of parameters $q = (\alpha,\beta)$ with the Euclidean norm. In this section we study the identification problem for the unknown parameters $q = (\alpha,\beta) \in \mathcal{P}$ in the problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \big(\alpha_0 + \alpha^2\big)\operatorname{div}\Big(\dfrac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) - \big(\beta_0 + \beta^2\big)\Delta\dfrac{\partial y}{\partial t} = f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\alpha_0,\beta_0 > 0$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, and $f \in L^2(0,T;H^{-1}(\Omega))$ are fixed. The pair of diffusive and viscosity coefficients $q = (\alpha,\beta)$ in (3.1) is an unknown parameter should be identified. In this setting we take the two-dimensional Euclidean space $\mathcal{P} = \mathbf{R}^2$ as the set of parameters (α,β). By Theorem 2.1 we have that for each $q \in \mathcal{P}$ there exists a unique weak solution $y = y(q) \in W(0,T)$ of (3.1).

First we shall show the continuous dependence of solutions on parameters $q = (\alpha,\beta)$.

Theorem 3.1. *The solution map $q \to y(q)$ from $\mathcal{P} = \mathbf{R}^2$ into $W(0,T)$ is continuous.*

Proof. Let $q = (\alpha, \beta)$ be arbitrarily fixed. Suppose $q_m = (\alpha_m, \beta_m) \to q = (\alpha, \beta)$ in $\mathcal{P}$. Let $y_m = y(q_m)$ and $y = y(q)$ be the solutions of (3.1) for $q = q_m$ and for q, respectively. Since $\{q_m\}$ is bounded in $\mathcal{P}$, by Theorem 2.1 and Remark 2.1, we see that

$$\left|y_m'(t)\right|^2 + \left|\nabla y_m(t)\right|^2 + \int_0^t \left|\nabla y_m'\right|^2 ds \leqslant C_0 < \infty, \quad \forall t \in [0,T], \tag{3.2}$$

where $C_0 > 0$ is a constant depending only on $\alpha_0, \beta_0, y_0, y_1$ and f. We set $z_m = y_m - y$. Then z_m satisfies

$$\begin{cases} \dfrac{\partial^2 z_m}{\partial t^2} - \left(\alpha_0 + \alpha_m^2\right)\operatorname{div}\left(G(\nabla y_m) - G(\nabla y)\right) - \left(\beta_0 + \beta_m^2\right)\Delta \dfrac{\partial z_m}{\partial t} \\ \quad = -\left(\alpha^2 - \alpha_m^2\right)\operatorname{div} G(\nabla y) - \left(\beta^2 - \beta_m^2\right)\Delta \dfrac{\partial y}{\partial t} \quad \text{in } Q, \\ z_m = 0 \quad \text{on } \Sigma, \\ z_m(0,x) = 0, \quad \dfrac{\partial z_m}{\partial t}(0,x) = 0 \quad \text{in } \Omega \end{cases} \tag{3.3}$$

in the weak sense. Multiply the weak form of (3.3) by z_m' and z_m, integrate them over $[0,t]$ and add the integrals, then we have

$$\begin{aligned} &\left|z_m'(t)\right|^2 + \left(\beta_0 + \beta_m^2\right)\left|\nabla z_m(t)\right|^2 + 2\left(\beta_0 + \beta_m^2\right)\int_0^t \left|\nabla z_m'\right|^2 ds \\ &= \int_0^t \left(2\left(\alpha^2 - \alpha_m^2\right)G(\nabla y) + 2\left(\beta^2 - \beta_m^2\right)\nabla y', \nabla z_m + \nabla z_m'\right) ds \\ &\quad - 2\left(\alpha_0 + \alpha_m^2\right)\int_0^t \left(G(\nabla y_m) - G(\nabla y), \nabla z_m + \nabla z_m'\right) ds \\ &\quad - 2\left(z_m'(t), z_m(t)\right) + 2\int_0^t \left|z_m'\right|^2 ds. \end{aligned} \tag{3.4}$$

We estimate the right-hand side of (3.4). We put

$$\Phi_m(t) = \int_0^t \left(2\left(\alpha^2 - \alpha_m^2\right)G(\nabla y) + 2\left(\beta^2 - \beta_m^2\right)\nabla y', \nabla z_m + \nabla z_m'\right) ds. \tag{3.5}$$

Since $|z_m| \leqslant |y| + |y_m|$, we can verify by (2.2), (3.2) and Schwartz inequality that

$$\left|\Phi_m(t)\right| \leqslant K_0\left(\left|\alpha^2 - \alpha_m^2\right| + \left|\beta^2 - \beta_m^2\right|\right), \quad \forall t \in [0,T], \tag{3.6}$$

for some $K_0 > 0$. By (2.2), we see

$$\left|G(\nabla y_m) - G(\nabla y)\right| \leqslant 2|\nabla z_m|. \tag{3.7}$$

We notice here that $\alpha_0 + \alpha_m^2$ is uniformly bounded in m. Let $\epsilon > 0$ be an arbitrary number. Then, by (3.7) and Schwartz inequality, we can deduce

$$2\left|\left(\alpha_0 + \alpha_m^2\right)\int_0^t \left(G(\nabla y_m) - G(\nabla y), \nabla z_m + \nabla z_m'\right) ds\right| \leqslant K_1(\epsilon)\int_0^t |\nabla z_m|^2 ds + \epsilon\int_0^t \left|\nabla z_m'\right|^2 ds, \tag{3.8}$$

where $K_1(\epsilon)$ is a positive constant depending on ϵ. We choose $\epsilon = \beta_0 > 0$. Then by (3.4), (3.8) and the positivity of β_0, we can obtain

$$\left|z_m'(t)\right|^2 + \left|\nabla z_m(t)\right|^2 + \int_0^t \left|\nabla z_m'\right|^2 ds \leqslant C_1 \Phi_m(t) + C_2 \int_0^t \left(\left|z_m'\right|^2 + |\nabla z_m|^2\right) ds, \tag{3.9}$$

where $C_1, C_2 > 0$. Hence applying Gronwall's inequality to (3.9), we have

$$\left|z_m'(t)\right|^2 + \left|\nabla z_m(t)\right|^2 + \int_0^t \left|\nabla z_m'\right|^2 ds \leqslant C_1 \Phi_m(t) + C_1 C_2 \exp(C_2 T) \int_0^t \Phi_m \, ds. \tag{3.10}$$

Since $\Phi_m(t) \to 0$ as $m \to \infty$ for all $t \in [0, T]$ by (3.6), we obtain from (3.10) that

$$\begin{aligned} &z_m(\cdot) \to 0 \quad \text{in } C\big([0,T]; H_0^1(\Omega)\big), \\ &z_m'(\cdot) \to 0 \quad \text{in } C\big([0,T]; L^2(\Omega)\big) \cap L^2\big(0,T; H_0^1(\Omega)\big), \end{aligned}$$

so that

$$y_m(\cdot) \to y(\cdot) \quad \text{strongly in } W(0,T).$$

This proves Theorem 3.1. □

The cost functionals given in the Introduction are represented by

$$J_1(q) = \left\| y(q) - z_{d_1} \right\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{for } q \in \mathcal{P}, \tag{3.11}$$

$$J_2(q) = \left\| y'(q) - z_{d_2} \right\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{for } q \in \mathcal{P}, \tag{3.12}$$

respectively, where $z_{d_i} \in L^2(0,T; L^2(\Omega))$, $i = 1, 2$. We choose the cost $J = J_1$ or $J = J_2$ for the identification of $q = (\alpha, \beta)$.

Assume that an admissible subset $\mathcal{P}_{\mathrm{ad}}$ of $\mathcal{P}$ is convex and closed. If $\mathcal{P}_{\mathrm{ad}}$ is compact, then for the minimizing sequence $\{q_m\}$ such as $J(q_m) \to J^* = \inf\{J(q)\colon q \in \mathcal{P}_{\mathrm{ad}}\}$ we can choose a subsequence $\{q_{mj}\}$ of $\{q_m\}$ such that $q_{mj} \to q^* \in \mathcal{P}_{\mathrm{ad}}$ and $y(q_{mj}) \to y(q^*)$ strongly in $W(0,T)$ by Theorem 3.1. Due to the continuous imbedding $W(0,T) \hookrightarrow C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ we have $J^* = J(q^*)$ for the costs (3.11) and (3.12). Then we have the following corollary.

Corollary 3.1. *If $\mathcal{P}_{\mathrm{ad}}$ is compact, then there exists at least one optimal parameter $q^* \in \mathcal{P}_{\mathrm{ad}}$ for the cost J_1 in* (3.11) *or J_2 in* (3.12).

Let the cost $J = J(q)$ be given over $\mathcal{P}$. Let the admissible set $\mathcal{P}_{\mathrm{ad}}$ be closed and convex in $\mathcal{P}$ and let $q^* = (\alpha^*, \beta^*)$ be an optimal parameter on $\mathcal{P}_{\mathrm{ad}}$ for the cost J. As is well known the necessary optimality condition for the cost J is given by

$$DJ(q^*)(q - q^*) \geqslant 0 \quad \text{for all } q \in \mathcal{P}_{\mathrm{ad}}, \tag{3.13}$$

where $DJ(q^*)$ denotes the Gâteaux derivative of $J(q)$ at $q = q^*$.

If the cost $J(q)$ is given by a quadratic functional $J(q) = J(y(q))$ over the solution space $W(0,T)$, then the Gâteaux differentiability of $J(q)$ follows from that of $y(q)$ in q. We can prove that the solution map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is Gâteaux differentiable. The following theorem gives the characterization of the Gâteaux derivatives as in [5].

Theorem 3.2. *The map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is Gâteaux differentiable at $q = q^*$ and such a Gâteaux derivative of $y(q)$ at $q = q^*$ in the direction $q - q^* \in \mathcal{P}$, say $z = Dy(q^*)(q - q^*)$, is a unique weak solution of the following linear problem*

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - (\alpha_0 + \alpha^{*2}) \operatorname{div}\left(\dfrac{\nabla z}{\sqrt{1 + |\nabla y^*|^2}} - \dfrac{\nabla y^* \cdot \nabla z}{(1 + |\nabla y^*|^2)^{\frac{3}{2}}} \nabla y^* \right) - (\beta_0 + \beta^{*2}) \Delta \dfrac{\partial z}{\partial t} \\ \quad = 2\alpha^*(\alpha - \alpha^*) \operatorname{div} G(\nabla y^*) + 2\beta^*(\beta - \beta^*) \Delta \dfrac{\partial y^*}{\partial t} \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(0,x) = 0, \quad \dfrac{\partial z}{\partial t}(0,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{3.14}$$

where $y^* = y(q^*)$.

Proof. Let $\lambda \in [-1, 1]$, and let y_λ and y^* be the weak solutions of (3.1) corresponding to $q^* + \lambda(q - q^*)$ and q^*, respectively. We set $z_\lambda = \lambda^{-1}(y_\lambda - y^*)$, $\lambda \neq 0$. Then z_λ is a solution of the following problem in the weak sense:

$$\begin{cases} \dfrac{\partial^2 z_\lambda}{\partial t^2} - (\alpha_0 + \alpha^{*2}) \operatorname{div} \dfrac{1}{\lambda}\big(G(\nabla y_\lambda) - G(\nabla y^*)\big) - (\beta_0 + \beta^{*2}) \Delta \dfrac{\partial z_\lambda}{\partial t} = F_\lambda \quad \text{in } Q, \\ z_\lambda = 0 \quad \text{on } \Sigma, \\ z_\lambda(0,x) = 0, \quad \dfrac{\partial z_\lambda}{\partial t}(0,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{3.15}$$

where

$$F_\lambda = \big(2\alpha^*(\alpha - \alpha^*) + \lambda(\alpha - \alpha^*)^2\big) \operatorname{div} G(\nabla y_\lambda) + \big(2\beta^*(\beta - \beta^*) + \lambda(\beta - \beta^*)^2\big) \Delta \frac{\partial y_\lambda}{\partial t}. \tag{3.16}$$

It is verified by (2.4) and Remark 2.1 that $F_\lambda \in L^2(0,T; H^{-1}(\Omega))$ and

$$\|F_\lambda\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C(\lambda) \|y_\lambda\|_{W(0,T)} \leqslant C_3 < \infty, \tag{3.17}$$

where C_3 is a constant independent of λ. We set

$$\mathcal{G}_\lambda = \frac{1}{\lambda}\big(G(\nabla y_\lambda) - G(\nabla y^*)\big). \tag{3.18}$$

Then we see easily by (2.2) that

$$|\mathcal{G}_\lambda| \leqslant 2|\nabla z_\lambda|. \tag{3.19}$$

Hence, by repeating similar calculations using Gronwall's inequality as in the proof of Theorem 3.1, we can deduce

$$\big|\nabla z_\lambda(t)\big|^2 + \big|z'_\lambda(t)\big|^2 + \int_0^t \big|\nabla z'_\lambda\big|^2 ds \leqslant K_2 \|F_\lambda\|^2_{L^2(0,T;H^{-1}(\Omega))} \leqslant K_2 C_3 \tag{3.20}$$

for some $K_1 > 0$. Therefore there exists $z \in W(0,T)$ and a sequence $\{\lambda_k\} \subset [-1, 1]$ tending to 0 such that

$$\begin{cases} z_{\lambda_k} \to z \quad \text{weakly star in } L^\infty\big(0,T; H_0^1(\Omega)\big) \text{ and weakly in } L^2\big(0,T; H_0^1(\Omega)\big) \text{ as } k \to \infty, \\ z'_{\lambda_k} \to z' \quad \text{weakly star in } L^\infty\big(0,T; L^2(\Omega)\big) \text{ and weakly in } L^2\big(0,T; H_0^1(\Omega)\big) \text{ as } k \to \infty, \\ z(0) = 0, \quad z'(0) = 0. \end{cases} \tag{3.21}$$

By the strong convergence $y_\lambda \to y^*$ in $W(0,T)$ as $\lambda \to 0$ and (3.21), we can prove as in [5] that

$$\operatorname{div} \mathcal{G}_{\lambda_k} \to \operatorname{div}\left(\frac{\nabla z}{\sqrt{1 + |\nabla y^*|^2}} - \frac{\nabla y^* \cdot \nabla z}{(1 + |\nabla y^*|^2)^{\frac{3}{2}}} \nabla y^* \right) \quad \text{weakly in } L^2\big(0,T; H^{-1}(\Omega)\big) \tag{3.22}$$

as $\lambda_k \to 0$. At the same time we can prove that

$$F_\lambda \to F \equiv 2\alpha^*(\alpha - \alpha^*) \operatorname{div} G(\nabla y^*) + 2\beta^*(\beta - \beta^*) \Delta \frac{\partial y^*}{\partial t} \quad \text{strongly in } L^2\big(0,T; H^{-1}(\Omega)\big). \tag{3.23}$$

Consequently from (3.21), (3.22) and (3.23), z is a unique weak solution satisfying (3.14). Hence, by the uniqueness of solutions of (3.14), $z_\lambda \to z$ weakly in $W(0,T)$. This proves that the map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is weakly Gâteaux differentiable and the Gâteaux derivative of $y(q)$ at $q = q^*$ in the direction $q - q^* \in \mathcal{P}$ is given by the unique solution z of (3.14). Further, we can prove the strong convergence of $\{z_\lambda\}$ to z in $W(0,T)$. The proof is quite similar, by adding the forcing term $F_\lambda - F$ in the equations for $z_\lambda - z$, to that of [5, pp. 336–337]. This completes the proof. $\square$

Now we put

$$\mathcal{F}(q-q^*;y^*) = \left(2\alpha^*(\alpha-\alpha^*)G(\nabla y^*) + 2\beta^*(\beta-\beta^*)\nabla\frac{\partial y^*}{\partial t}\right) \tag{3.24}$$

and

$$\mathcal{G}(\phi,\psi) = \left(\frac{\nabla\phi}{\sqrt{1+|\nabla\psi|^2}} - \frac{\nabla\psi\cdot\nabla\phi}{(1+|\nabla\psi|^2)^{\frac{3}{2}}}\nabla\psi\right), \quad \forall\phi,\psi \in H_0^1(\Omega). \tag{3.25}$$

Note that $F = \operatorname{div}\mathcal{F}(q-q^*;y^*)$ in (3.23) and $\mathcal{G}(\phi,\psi)$ is linear in ϕ for fixed ψ.

3.1. Case of distributive observations

The cost functional J_1 in (3.11) is represented by

$$J_1(q) = \int_0^T \left|y(q;t) - Y_1(t)\right|^2 dt, \quad q \in \mathcal{P}. \tag{3.26}$$

Then it is easily verified that the optimality condition (3.13) is written as

$$\int_0^T \left(y(q^*;t) - Y_1(t), Dy(q^*)(q-q^*)(t)\right) dt \geqslant 0, \quad \forall q \in \mathcal{P}_{\mathrm{ad}}, \tag{3.27}$$

where $q^* = (\alpha^*,\beta^*)$ is the optimal parameter for (3.26) and $z = Dy(q^*)(q-q^*)$ is a weak solution of (3.14).

Theorem 3.3. *The optimal parameter $q^* = (\alpha^*,\beta^*)$ for* (3.26) *is characterized by the following system of equations and inequality*:

$$\begin{cases} \dfrac{\partial^2 y^*}{\partial t^2} - \left(\alpha_0 + \alpha^{*2}\right)\operatorname{div}\left(\dfrac{\nabla y^*}{\sqrt{1+|\nabla y^*|^2}}\right) - \left(\beta_0+\beta^{*2}\right)\Delta\dfrac{\partial y^*}{\partial t} = f \quad \text{in } Q, \\ y^* = 0 \quad \text{on } \Sigma, \\ y^*(0,x) = y_0(x), \quad \dfrac{\partial y^*}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} - \left(\alpha_0+\alpha^{*2}\right)\operatorname{div}\left(\dfrac{\nabla p}{\sqrt{1+|\nabla y^*|^2}} - \dfrac{\nabla y^*\cdot\nabla p}{(1+|\nabla y^*|^2)^{\frac{3}{2}}}\nabla y^*\right) + \left(\beta_0+\beta^{*2}\right)\Delta\dfrac{\partial p}{\partial t} = y^* - Y_1 \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(T,x) = 0, \quad \dfrac{\partial p}{\partial t}(T,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{3.28}$$

$$\int_Q \nabla p\cdot\left(2(\alpha-\alpha^*)\alpha^* G(\nabla y^*) + 2(\beta-\beta^*)\beta^*\nabla\frac{\partial y^*}{\partial t}\right) dx\,dt \leqslant 0, \quad \forall q = (\alpha,\beta) \in \mathcal{P}_{\mathrm{ad}}. \tag{3.29}$$

Proof. Since $y^* - Y_1 \in L^2(0,T;L^2(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$, it is verified by the time reversion $t \to T-t$ via [2, p. 558] there is a unique weak solution $p \in W(0,T)$ of (3.28). We shall show (3.29). We proceed the calculations in the Gelfand triple space $(H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega))$ as in [5]. Multiplying both sides of the weak form of (3.28) by $z = Dy(q^*)(q-q^*)$ and integrating it by parts on $[0,T]$, we have that

$$\begin{aligned}\int_0^T (y^* - Y_1, z)\,dt &= \int_0^T \langle p'' - A^* \operatorname{div}\mathcal{G}(p, y^*) + B^*\Delta p', z\rangle\,dt \\ &= \int_0^T \langle p, z'' - A^* \operatorname{div}\mathcal{G}(z, y^*) - B^*\Delta z'\rangle\,dt \\ &= \int_0^T \langle p, \operatorname{div}\mathcal{F}(q - q^*; y^*)\rangle\,dt \\ &= -\int_0^T \big(\nabla p, \mathcal{F}(q - q^*; y^*)\big)\,dt, \end{aligned} \tag{3.30}$$

where $A^* = \alpha_0 + \alpha^{*2}$ and $B^* = \beta_0 + \beta^{*2}$. Therefore, (3.30) and (3.27) imply that the required optimality condition is given by (3.29). This proves Theorem 3.3. □

3.2. *Case of velocity observations*

The cost functional J_2 in (3.12) is represented by

$$J_2(q) = \int_0^T \big|y'(q;t) - Y_2(t)\big|^2\,dt, \quad q \in \mathcal{P}. \tag{3.31}$$

The optimality condition (3.13) for (3.31) is given by

$$\int_0^T \big(y'(q^*;t) - Y_2(t), Dy(q^*)(q - q^*)'(t)\big)\,dt \geqslant 0, \quad \forall q \in \mathcal{P}_{\mathrm{ad}}, \tag{3.32}$$

where $z = Dy(q^*)(q - q^*)$ is a weak solution of (3.14). As indicated in [5], we introduce an adjoint system represented by the following first order integro-differential equation

$$\begin{cases} \dfrac{\partial p}{\partial t} + \big(\alpha_0 + \alpha^{*2}\big)\displaystyle\int_t^T \operatorname{div}\mathcal{G}\big(p(s), y^*(s)\big)\,ds + \big(\beta_0 + \beta^{*2}\big)\Delta p = \dfrac{\partial y^*}{\partial t} - Y_2 \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(T, x) = 0 \quad \text{in } \Omega. \end{cases} \tag{3.33}$$

Since $\frac{\partial y^*}{\partial t} - Y_2 \in L^2(Q) = L^2(0, T; L^2(\Omega))$, by reversing the direction of time $t \to T - t$ and applying the result of [2, pp. 656–662] to the problem (3.33), we have a unique weak solution p of (3.33) satisfying

$$p \in W\big(H_0^1(\Omega), L^2(\Omega)\big) \cap C\big([0, T]; H_0^1(\Omega)\big). \tag{3.34}$$

Theorem 3.4. *The optimal parameter $q^* = (\alpha^*, \beta^*)$ for* (3.31) *is characterized by the following system of equations and inequality*:

$$\begin{cases} \dfrac{\partial^2 y^*}{\partial t^2} - \big(\alpha_0 + \alpha^{*2}\big)\operatorname{div}\bigg(\dfrac{\nabla y^*}{\sqrt{1 + |\nabla y^*|^2}}\bigg) - \big(\beta_0 + \beta^{*2}\big)\Delta\dfrac{\partial y^*}{\partial t} = f \quad \textit{in } Q, \\ y^* = 0 \quad \textit{on } \Sigma, \\ y^*(0, x) = y_0(x), \quad \dfrac{\partial y^*}{\partial t}(0, x) = y_1(x) \quad \textit{in } \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial p}{\partial t} + \left(\alpha_0 + \alpha^{*2}\right) \displaystyle\int_t^T \operatorname{div}\left(\frac{\nabla p}{\sqrt{1+|\nabla y^*|^2}} - \frac{\nabla y^* \cdot \nabla p}{(1+|\nabla y^*|^2)^{\frac{3}{2}}} \nabla y^* \right) ds + \left(\beta_0 + \beta^{*2}\right) \Delta p \\ \quad = \dfrac{\partial y^*}{\partial t} - Y_2 \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(T,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{3.35}$$

$$\int_Q \nabla p \cdot \left(2(\alpha - \alpha^*)\alpha^* G(\nabla y^*) + 2(\beta - \beta^*)\beta^* \nabla \frac{\partial y^*}{\partial t} \right) dx\, dt \geqslant 0, \quad \forall q = (\alpha, \beta) \in \mathcal{P}_{\mathrm{ad}}. \tag{3.36}$$

Proof. Multiplying both sides of the weak form of (3.35) by $z' = Dy(q^*)(q - q^*)'$, taking dual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and integrating it by parts on $[0, T]$, we have that

$$\begin{aligned} \int_0^T (y^{*\prime} - Y_2, z')\, dt &= \int_0^T \left\langle p' + A^* \int_t^T \operatorname{div} \mathcal{G}\big(p(s), y^*(s)\big)\, ds + B^* \Delta p, z' \right\rangle dt \\ &= -\int_0^T \big\langle p, z'' - A^* \operatorname{div} \mathcal{G}\big(p(t), y^*(t)\big) - B^* \Delta z' \big\rangle\, dt \\ &= -\int_0^T \big\langle p, \operatorname{div} \mathcal{F}(q - q^*; y^*) \big\rangle\, dt = \int_0^T \big(\nabla p, \mathcal{F}(q - q^*; y^*)\big)\, dt, \end{aligned} \tag{3.37}$$

where $A^* = \alpha_0 + \alpha^{*2}$ and $B^* = \beta_0 + \beta^{*2}$. Hence, (3.37) and (3.32) imply that the required optimality condition is given by (3.36). $\square$

Let us deduce the bang-bang principle from (3.36) for the case where $\mathcal{P}_{\mathrm{ad}}$ is given by $\mathcal{P}_{\mathrm{ad}} = [0, \alpha_1] \times [0, \beta_1]$. In this case the necessary condition (3.36) is equivalent to

$$\alpha^*(\alpha - \alpha^*) \int_Q \nabla p \cdot G(\nabla y^*)\, dx\, dt \geqslant 0, \quad \forall \alpha \in [0, \alpha_1], \tag{3.38}$$

$$\beta^*(\beta - \beta^*) \int_Q \nabla p \cdot \nabla \frac{\partial y^*}{\partial t}\, dx\, dt \geqslant 0, \quad \forall \beta \in [0, \beta_1]. \tag{3.39}$$

First we consider (3.38). Put $a = \int_Q \nabla p \cdot G(\nabla y^*)\, dx\, dt$ and assume that $a \neq 0$. Then (3.38) is rewritten simply by

$$\alpha^*(\alpha - \alpha^*) a \geqslant 0, \quad \forall \alpha \in [0, \alpha_1].$$

It is easily verified from the above inequality that α^* is given by

$$\alpha^* = -\frac{1}{2}\big\{\operatorname{sign}(a) - 1\big\}\alpha_1 \quad \text{or} \quad \alpha^* = 0.$$

Similarly, if $\int_Q \nabla p \cdot \nabla \frac{\partial y^*}{\partial t}\, dx\, dt = b (\neq 0)$, then

$$\beta^* = -\frac{1}{2}\big\{\operatorname{sign}(b) - 1\big\}\beta_1 \quad \text{or} \quad \beta^* = 0.$$

These are the so-called bang-bang principle for the optimal parameter $q^* = (\alpha^*, \beta^*)$.

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